Partial Gauss decomposition, $U_q(\widehat{\mathfrak{gl}(n-1)}) \in U_q(\widehat{\mathfrak{gl}(n)})$ and Zamolodchikov algebra

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Partial Gauss decomposition, $U_q(\widehat{\mathfrak{gl}(n-1)}) \in U_q(\widehat{\mathfrak{gl}(n)})$ and Zamolodchikov algebra

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Abstract. We use the idea of partial Gauss decomposition to study structures related to $U_q(\widehat{\mathfrak{gl}(n-1)})$ inside $U_q(\widehat{\mathfrak{gl}(n)})$. This gives a description of $U_q(\widehat{\mathfrak{gl}(n)})$ as an extension of $U_q(\widehat{\mathfrak{gl}(n-1)})$ with Zamolodchikov algebras. We describe the connection of this new realization with form factors.

1. Introduction

The affine Kac-Moody algebra $\hat{\mathfrak{g}}$, associated to a simple Lie algebra $\mathfrak{g}$, admits a natural realization as a central extension of the corresponding loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. Drinfeld gives a similar realization for $U_q(\hat{\mathfrak{g}})$, which is called the Drinfeld realization [D1]. Faddeev, Reshetikhin, Takhtajan and Semenov-Tian-Shansky [FRT, RS] present a realization of $U_q(\mathfrak{g})$ to the quantum loop algebra $U_q(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}])$ using a solution of the Yang–Baxter equation depending on a parameter $z \in \mathbb{C}$

$$R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z)$$

where $R(z)$ is a rational function of z with values in $\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$. An explicit identification between the two realizations of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ for the case $\mathfrak{g} = \mathfrak{gl}(\mathfrak{n})$ is established [DF] by applying Gauss decomposition to the L-operators for the FRTS realization.

In this paper, we will use the idea of partial Gauss decomposition to study the structures related to $U_q(\widehat{\mathfrak{gl}(n-1)})$ inside $U_q(\widehat{\mathfrak{gl}(n)})$. We show that $U_q(\widehat{\mathfrak{gl}(n)})$ can be described as an extension of $U_q(\widehat{\mathfrak{gl}(n-1)})$ with Zamolodchikov algebras, where the Zamolodchikov algebras can be interpreted as certain intertwiners for $U_q(\widehat{\mathfrak{gl}(n-1)})$. The Zamolodchikov algebras are used to derive structures related to form factors and related structures.

This paper uses partial Gauss decomposition in order to find new structures hidden inside the affine quantum groups. This method is related to many aspects of the theory of affine quantum groups [Di2, MJ, M] and physics [Sm]. The meaning of this method can be explained using the method of the twisting of Drinfeld [D2, KT].

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2. Quantum algebra $U_q(\hat{\mathfrak{gl}}(n-1)) \in U_q(\hat{\mathfrak{gl}}(n))$ and partial Gauss decomposition

Let V be $\mathbb{C}^n$ with a fixed basis e_i, $i = 1, \ldots, n$ and E_{ij} be the standard basis of $\mathrm{End}(\mathbb{C}^n)$ corresponding to e_i. Let $R(z)$ be an element of $\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ defined by

$$R(z) = \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \sum_{\substack{i \neq j \\ i,j=1}}^{n} E_{ii} \otimes E_{jj} \frac{z-1}{q^{-1}z - q} + \sum_{\substack{i > j \\ i,j=1}}^{n} E_{ij} \otimes E_{ji} \frac{(q^{-1}-q)}{zq^{-1}-q}$$
$$+ \sum_{\substack{i < j \\ i,j=1}}^{n} E_{ij} \otimes E_{ji} \frac{z(q^{-1}-q)}{zq^{-1}-q}$$

where q, z are formal variables. Then $R(z)$ satisfies the Yang–Baxter equation and R is unitary, namely

$$R_{21}(z)^{-1} = R(z^{-1})$$

where $R_{21}(z) = PR_{12}(z)P$, where P is the operator permuting the two components of $V \otimes V$.

Faddeev, Reshetikhin and Takhtajan defined a Hopf algebra structure using $R(z)$, which satisfies the Yang–Baxter equation. Reshetikhin and Semenov-Tian-Shansky obtained a central extension of this algebra. The algebra defined with the $R(z)$ above is isomorphic to $U_q(\hat{\mathfrak{gl}}(n))$. The central extension is incorporated in shifts of the parameter z in $R(z)$.

Definition 2.1. *$U_q(\widehat{\mathfrak{gl}}(n))$ is an associative algebra with generators $\{l_{ij}^{\pm}[\mp m]$, $m \in \mathbf{Z}_+ \setminus \mathbf{0}, l_{ij}^{+}[0], l_{ji}^{-}[0]$, $1 \leqslant j \leqslant i \leqslant n\}$. Let $l_{ij}^{\pm}(z) = \sum_{m=0}^{\infty} l_{ij}^{\pm}[\pm m] z^{\pm m}$, where $l_{ij}^{+}[0] = l_{ji}^{-}[0] = 0$, for $1 \leqslant j > i \leqslant n$. Let $L^{\pm}(z) = (l_{ij}^{\pm}(z))_{i,j=1}^{n}$. Then the defining relations are the following:*

$$l_{ii}^{+}[0] l_{ii}^{-}[0] = l_{ii}^{-}[0] l_{ii}^{+}[0] = 1$$
$$R\left(\frac{z}{w}\right) L_1^{\pm}(z) L_2^{\pm}(w) = L_2^{\pm}(w) L_1^{\pm}(z) R\left(\frac{z}{w}\right)$$
$$R\left(\frac{z_-}{w_+}\right) L_1^{+}(z) L_2^{-}(w) = L_2^{-}(w) L_1^{+}(z) R\left(\frac{z_+}{w_-}\right)$$

where $z_{\pm} = zq^{\pm\frac{c}{2}}$. The expansion direction of $R(\frac{z}{w})$ is chosen to be in $\frac{z}{w}$ or $\frac{w}{z}$ respectively [DF].

The Hopf algebra is given by

$$\Delta L^{\pm}(z) = L^{\pm}(zq^{\pm(1\otimes\frac{c}{2})}) \dot{\otimes} L^{\pm}(zq^{\mp(\frac{c}{2}\otimes 1)})$$

which is defined as

$$\Delta(l_{ij}^{\pm}(z)) = \sum_{k=1}^{n} l_{ik}^{\pm}(zq^{\pm(1\otimes\frac{c}{2})}) \otimes l_{kj}^{\pm}(zq^{\mp(\frac{c}{2}\otimes 1)})$$

and its antipode is

$$S(L^{\pm}(z)) = L^{\pm}(z)^{-1}.$$

The invertibilty of $L^{\pm}(z)$ follows from the properties that $l_{ii}^{\pm}$ are invertible and $L^{\pm}(0)$ are upper triangular and lower triangular, respectively.

$L^{\pm}(z)$ have the following unique decompositions:

$$L^{\pm}(z) = \begin{pmatrix} 1 & & & & 0 \\ e^{\pm}_{2,1}(z) & \ddots & & & \\ e^{\pm}_{3,1}(z) & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ e^{\pm}_{n,1}(z) & \dots & e^{\pm}_{n,n-1}(z) & e^{\pm}_{n-1,n}(z) & 1 \end{pmatrix} \begin{pmatrix} k^{\pm}_1(z) & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & k^{\pm}_n(z) \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & f^{\pm}_{1,2}(z) & f^{\pm}_{1,3}(z) & \dots & f^{\pm}_{1,n}(z) \\ & \ddots & \ddots & \ddots & \vdots \\ & & & & f^{\pm}_{n-1,n}(z) \\ 0 & & & & 1 \end{pmatrix}.$$

These are used to establish the isomorphism between the Drinfeld realizations of $U_q(\widehat{\mathfrak{gl}(n)})$ and its FRTS realization.

Similarly, we have the following partial Gauss decomposition:

Proposition 2.1. *The operator $L^{\pm}(z)$ can be uniquely decomposed as*

$$L^{\pm}(z) = \begin{pmatrix} I & 0 \\ e^{\pm}(z) & 1 \end{pmatrix} \begin{pmatrix} K^{\pm}(z) & 0 \\ 0 & k^{\pm}(z) \end{pmatrix} \begin{pmatrix} I & f^{\pm}(z) \\ 0 & 1 \end{pmatrix}$$

where $K^{\pm}(z)$ and $k^{\pm}(z)$ are $(n-1)\times(n-1)$ invertible matrix operators, $e^{\pm}(z)$ is a column vector of size $(n-1)$ column and $f^{\pm}(z)$ is a row vector of size $(n-1)$.

Because $K^{\pm}(z)$ are invertible, the elements $e^{\pm}(z)$, $f^{\pm}(z)$ and $k^{\pm}(z)$ are uniquely expressed in terms of the matrix coefficients of $L^{\pm}(z)$.

We now have the following proposition.

Proposition 2.2. *The algebra generated by entries of operator matrices $K^{\pm}(z)$ is $U_q(\hat{\mathfrak{gl}}(n-1))$.*

We will follow the steps as shown in [DF] to find the complete commutation relations for the operators in the above decomposition. For the calculation, we need the following formulae:

$$L^{\pm}(z) = \begin{pmatrix} K^{\pm}(z) & K^{\pm}(z)f^{\pm}(z) \\ e^{\pm}(z)K^{\pm}(z) & (k^{\pm}(z) + e^{\pm}(z)K^{\pm}(z)f^{\pm}(z)) \end{pmatrix}$$

$$L_1(z)L_2(w) = \begin{pmatrix} K(z)K(w) & K(z)K(w)f(w) & K(z)f(z)K(w) & K(z)f(z)K(w)f(w) \\ K(z)e(w)K(w) & K(z)D(z) & K(z)f(z)e(w)K(w) & K(z)f(z)D(w) \\ e(z)K(z)K(w) & e(z)K(z)K(w)f(w) & D(z)K(w) & D(z)(K(w)f(w)) \\ e(z)K(z)e(w)K(w) & e(z)K(z)D(w) & D(z)e(w)K(w) & D(z)(D(w))K(w)f(w) \end{pmatrix}$$

$$(L^{\pm}(z))^{-1} = \begin{pmatrix} K^{\pm}(z)^{-1} + f^{\pm}(z)k^{\pm}(z)^{-1}e^{\pm}(z) & -f^{\pm}(z)k^{\pm}(z)^{-1} \\ -k^{\pm}(z)^{-1}e^{\pm}(z) & k^{\pm}(z)^{-1} \end{pmatrix}$$

$$R_{12}\left(\frac{z}{w}\right) = \begin{pmatrix} \bar{R}(z/w) & 0 & 0 & 0 \\ 0 & \frac{z-w}{zq^{-1}-wq}A & \frac{-z(q-q^{-1})}{zq^{-1}-wq}B & 0 \\ 0 & \frac{-w(q-q^{-1})}{zq^{-1}-wq}C & \frac{z-w}{zq^{-1}-wq}D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_{12}\left(\frac{z}{w}\right)^{-1} = R_{21}\left(\frac{w}{z}\right) = \begin{pmatrix} \bar{R}_{21}(w/z) & 0 & 0 & 0 \\ 0 & \frac{z-w}{zq-wq^{-1}}A & \frac{-z(q-q^{-1})}{wq^{-1}-zq}B & 0 \\ 0 & \frac{-w(q-q^{-1})}{wq^{-1}-zq}C & \frac{z-w}{zq-wq^{-1}}D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(L_1(w))^{-1} = \begin{pmatrix} * & 0 & -f(w)k(w)^{-1} & 0 \\ 0 & * & 0 & -f(w)k(w)^{-1} \\ -k(w)^{-1}e(w) & 0 & k(w)^{-1} & 0 \\ 0 & -k(w)^{-1}e(w) & 0 & k(w)^{-1} \end{pmatrix}$$

where $A = \sum_{i\neq n} E_{ii} \otimes E_{nn}$, $D = \sum_{i\neq n} E_{nn} \otimes E_{ii}$ $C = \sum_{j<n+1} E_{nj} \otimes E_{jn}$, $B = \sum_{j<n+1} E_{jn} \otimes E_{nj}$, $\bar{R}(z)$ is the R-matrix restricted to the subspace $V' \otimes V'$, V' is generated on the subspace generated by e_i, $i = 1, \ldots, n-1$, $D^{\pm}(z) = (k^{\pm}(z) + e^{\pm}(z)K^{\pm}(z)f^{\pm}(z))$; and

$$L_1^{\pm}(w)^{-1}R_{21}\left(\frac{z}{w}\right)L_2^{\pm}(z) = L_2^{\pm}(z)R_{21}\left(\frac{z}{w}\right)L_1^{\pm}(w)^{-1}$$

$$L_1^{-}(w)^{-1}R_{21}\left(\frac{z^{+}}{w^{-}}\right)L_2^{+}(z) = L_2^{+}(z)R_{21}\left(\frac{z_{-}}{w_{+}}\right)L_1^{-}(w)^{-1}$$

$$R_{21}\left(\frac{z_{-}}{w_{+}}\right)L_2^{-}(z)L_1^{+}(w) = L_1^{+}(w)L_2^{-}(z)R_{21}\left(\frac{z_{+}}{w_{-}}\right)$$

$$L_1^{+}(w)^{-1}R_{21}\left(\frac{z_{-}}{w_{+}}\right)L_2^{-}(z) = L_2^{-}(z)R_{21}\left(\frac{z_{+}}{w_{-}}\right)L_1^{+}(w)$$

$$L_2^{\pm}(z)^{-1}(L_1^{\pm}(w))^{-1}R\left(\frac{z}{w}\right) = R_{21}\left(\frac{z}{w}\right)(L_1^{\pm}(w))^{-1}(L_2^{\pm}(z))^{-1}$$

$$L_2^{+}(z)^{-1}L_1^{-}(w)^{-1}R_{21}\left(\frac{z_{+}}{w_{-}}\right) = R_{21}\left(\frac{z_{-}}{w_{+}}\right)(L_1^{-}(w))^{-1}(L_2^{+}(z))^{-1}.$$

Using the same calculation technique as in [DF], we have the following lemma.

Lemma 2.3.

$$\bar{R}(z/w)K_1^{\pm}(z)K_2^{\pm}(w) = K_2^{\pm}(w)K_1^{\pm}(z)\bar{R}(z/w)$$
$$k^{\pm}(z)k^{\pm}(w) = k^{\pm}(w)k^{\pm}(z)$$
$$\bar{R}(z_{+}/w_{-})K_1^{+}(z)K_2^{-}(w) = K_2^{-}(w)K_1^{+}(z)\bar{R}(z_{-}/w_{+})$$
$$k^{+}(z)k^{-}(w) = k^{-}(w)k^{+}(w)$$
$$k^{\pm}(z)k^{\pm}(w) = k^{\pm}(w)k^{\pm}(z)$$
$$\frac{z_{\mp}q^{-1} - w_{\pm}q}{z_{\mp} - w_{\pm}}k^{\mp}(w)^{-1}K^{\pm}(z) = K^{\pm}(z)k^{\mp}(w)\frac{z_{\pm}q^{-1} - w_{\mp}q}{z_{\pm} - w_{\mp}}$$
$$K_1^{\pm}(z)E_2(w) = \frac{zq^{\mp\frac{c}{2}-1} - wq}{zq^{\mp\frac{c}{2}} - w}E_2(w)\bar{R}(zq^{\mp\frac{c}{2}}/w)K_1^{\pm}(z)$$
$$K_1^{p}m(z)\bar{R}(zq^{\pm\frac{c}{2}}/w)F_2(w) = \frac{zq^{\pm\frac{c}{2}} - w}{zq^{\pm\frac{c}{2}-1} - wq}F_2(w)K_1^{\pm}(z)$$
$$k^{\pm}(z)E(w) = \frac{zq^{\mp\frac{c}{2}+1} - wq^{-1}}{zq^{\mp\frac{c}{2}} - w}E(w)k^{\pm}(z)$$
$$k^{\pm}(z)F(w) = \frac{zq^{\pm\frac{c}{2}} - w}{zq^{\pm\frac{c}{2}+1} - wq^{-1}}F(w)k^{\pm}(z)$$
$$(z - wq^2)E_1(z)E_2(w)R(z/w) = (zq^2 - w)E_2(w)E_1(z)$$
$$(zq^2 - w)F(z)F(w) = R(z/w)(z - wq^2)F(w)F(z)$$
$$E_2(z)(F_1(w)) - F_1(w)E_2(z) = (q - q^{-1})$$
$$\times\left(\delta\left(\frac{w}{z}q^{c}\right)k^{-}(wq^{\frac{c}{2}})K^{-}(wq^{\frac{c}{2}})^{-1} - \delta\left(\frac{w}{z}q^{-c}\right)k^{+}(wq^{-\frac{c}{2}})K^{+}(wq^{-\frac{c}{2}})^{-1}\right)$$

where $E(z) = e^{+}(zq^{\frac{c}{2}}) - e^{-}(zq^{\frac{-c}{2}})$, $F(z) = f^{+}(zq^{\frac{-c}{2}}) - f^{-}(zq^{\frac{c}{2}})$ *and* $\delta(x) = \sum_{m\in\mathbb{Z}} x^m$.

The algebra generated by $E(z)$ or $F(z)$ gives a realization of the Zamolodchikov algebra, the formulation above is basically the same as in [Di1], where we study the Hopf algebra extension of Zamolodchikov algebras. On the other hand, we can reformulate the definition of $U_q(\hat{\mathfrak{gl}(\mathfrak{n})})$ using the relations above.

Definition 2.2. *Let ZUR(n) be an algebra generated by matrix operators $E(z)$, $F(z)$, $K^{\pm}(z)$ and $k^{\pm}(z)$ associated with the vector space $V = \mathbb{C}^{n-1}$ respectively to V^*, V, $V \otimes V^*$ and a one-dimensional space $\mathbb{C}$. The commutation relations are defined as in the lemma above.*

Theorem 2.4. *ZUR(n) is isomorphic to $U_q(\hat{\mathfrak{gl}}(n))$.*

The proof follows the form of the above lemma and the similar argument in [DF].

From the point of view of [Di1], we can give a new Hopf algebra structure to this formulation using the similar formulae. The important point is that from the definition we can see that $E(z)$ and $F(z)$ are nothing but the intertwiners for the affine algebra $U_q(\hat{\mathfrak{gl}}(n-1))$ generated by the operators $K^{\pm}(z)(k^{\pm}(z))^{-1}$. The last formula of the commutation relations implies constructions as in [M, Sm].

Let $\bar{E}(z) = E(z)K^-(zq^{c/2}k^-(zq^{c/2})$, then we have the following.

Proposition 2.5.

$$E_2(z)\frac{(1-z/wq^{-c})R(z/w)(z-wq^2)}{zq^2-w}(F_1(w)) - F_1(w)E_2(z)$$
$$= (q-q^{-1})(1-q^{-2c})\delta\left(\frac{w}{z}q^{-c}\right)k^+(wq^{-\frac{c}{2}})K^+(wq^{-\frac{c}{2}})^{-1}k^-(wq^{\frac{c}{2}})^{-1}K^-(wq^{\frac{c}{2}})$$
$$(zq^2-w)E_1(z)E_2(w) = (z-wq^2)E_2(w)E_1(z)R(z/w)$$
$$(zq^2-w)F(z)_1F_2(w) = R(z/w)(z-wq^2)F_2(w)F_1(z)$$
$$(1-z/wq^cE_2(z)R(z/w)\frac{z-wq^2}{zq^2-w}(F_1(w)) - F_1(w)E_2(z) = (q-q^{-1})(1-q^{2c})\delta\left(\frac{w}{z}q^c\right).$$

The first formula above coincides with the spinor constructions of affine quantum groups in [Di2].

The last three formulae above say that these operators generate an algebra which is almost the same as the Zamolodchikov–Faddeev algebra used to describe the theory of form factors [Sm]. Similarly we can also define a new operator $\bar{F}(z) = (k^+(wq^{-\frac{c}{2}})^{-1}K^+(wq^{-\frac{c}{2}})F(z)$. This operator with $E(z)$ generates another algebra similar to the definition above. From the point of view of intertwiners as in [MJ], these operators can give a complete theory of form factors, where one copy of the algebra is explained as the Zamolodchikov–Faddeev algebra to define one model, and the other one is explained as a local operator, which commutes with the first algebra up to certain functions. In a subsequent paper we will apply the same method to Yangian and elliptic algebra [LKP, F], and give complete details to description of the more general Zamolodchikov–Faddeev type of algebras, whose degeneration gives us the corresponding results in this paper.

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